nD Cauchy problem.

$$\begin{cases} u_{tt} = \Delta u & \text{in } \mathbb{R}^n \times [0, T) \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n \\ u_t(x, 0) = h(x) & \end{cases}$$

$u \in C^2(\mathbb{R}^n \times [0, T))$

$g \in C^3(\mathbb{R}^n)$, $h \in C^2(\mathbb{R}^n)$.

* spherical means.

$x \in \mathbb{R}^n$, $r > 0$, $t \in [0, T)$, Define

$$U(x; r, t) = \fint_{\partial B_r(x)} u(\sigma, t)\, d\sigma$$

Similarly, Define

$$G(x; r) = \fint_{\partial B_r(x)} g(\sigma)\, d\sigma, \quad H(x; r) = \fint_{\partial B_r(x)} h(\sigma)\, d\sigma.$$

Lemma) Euler-Poisson-Darboux equation

Given a fixed $x \in \mathbb{R}^n$, we have

$$U_{tt} = U_{rr} + \frac{n-1}{r} U_r \quad \text{in } \mathbb{R}_+ \times (0, T)$$

$\mathbb{R}_+ = (0, \infty)$

$$U(x; r, 0) = G(x; r)$$
$$U_t(x; r, 0) = H(x; r)$$
on $\mathbb{R}_+$.

$$U \in C^2([0, \infty) \times [0, T))$$

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proof) $$U(x; r, t) = ⨍_{\partial B_r(x)} u(6, t) \, d6$$

$$= ⨍_{\partial B_1(0)} u(x + r6, t) \, d6$$

$$U_r(x; r, t) = ⨍_{\partial B_1(0)} u_\nu(x + r6, t) \, d6$$

$$= ⨍_{\partial B_r(x)} u_\nu(6, t) \, d6.$$

$$U_r = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u_r(\sigma, t)\, d\sigma$$

$$= \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta u(y, t)\, dy$$

$$= \frac{r}{n} \fint_{B_r(x)} \Delta u(y, t)\, dy$$

$$U_{rr} = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} \Delta u(\sigma, t)\, d\sigma$$

$$- \frac{n-1}{\omega_n r^n} \int_{B_r(x)} \Delta u(y, t)\, dy$$

$$= \fint_{\partial B_r(x)} \Delta u(\sigma, t)\, d\sigma - \frac{n-1}{n} \fint_{B_r(x)} \Delta u(y, t)\, dy$$

Since $u \in C^0$, $U \in C^0(\overline{\mathbb{R}_+} \times [0,T))$

and $\lim_{r \to 0^+} U(x;r,t) = u(x,t)$

Since $u \in C^2$, $\Delta u \in C^0$.

we have $U \in C^2(\overline{\mathbb{R}_+} \times [0,T))$,

and $\lim_{r \to 0^+} U_r(x;r,t) = 0$

$$\lim_{r \to 0^+} U_{rr}(x;r,t) = \frac{1}{n} \Delta u(x,t)$$

Next,

$$U_r = \frac{r}{n} \fint_{B_r(x)} \Delta u = \frac{1}{w_n r^{n-1}} \int_{B_r(x)} u_{tt}$$

$$\Rightarrow r^{n-1} U_r = \frac{1}{w_n} \int_{B_r(x)} u_{tt}(y,t)\,dy$$

$$\Rightarrow (r^{n-1} U_r)_r = \frac{1}{w_n} \int_{\partial B_r(x)} u_{tt}(\sigma,t)\,d\sigma$$

$$= r^{n-1} \fint_{\partial B_r(x)} u_{tt}(\sigma,t)\,d\sigma$$

$$= r^{n-1} \frac{\partial^2}{\partial t^2} \fint_{\partial B_r(x)} u(\sigma,t)\,d\sigma$$

$$= r^{n-1} U_{tt}.$$

$$\Rightarrow U_{tt} = U_{rr} + \frac{n-1}{r} U_r. \quad \blacksquare$$

Thm) [Kirchhoff's formula]

$(n=3)$ $g \in C^3(\mathbb{R}^3)$, $h \in C^2(\mathbb{R}^3)$

Then, $\exists$ a unique $u \in C^2(\mathbb{R}^3 \times [0,T))$

s.t. $\Delta u = u_{tt}$ in $\mathbb{R}^3 \times [0,T)$

$u = g$, $u_t = h$ on $\mathbb{R}^3 \times \{0\}$.

Moreover,

$$u(x,t) = \fint_{\partial B_t(x)} t h(\sigma) + g(\sigma) + t g_\nu(\sigma) \, d\sigma$$

$$= \fint_{\partial B_t(x)} t h(\sigma) + g(\sigma) + \nabla g(\sigma) \cdot (\sigma - x) \, d\sigma$$

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pf) Define $\hat{U}(x;r,t) = r U(x;r,t)$

$$\hat{G} = r G, \quad \hat{H} = r H.$$

$\Rightarrow$ $\hat{U}(x;r,0) = \hat{G}(x;r)$, $\hat{U}(x;0,t) = 0$.

$\hat{U}_t(x;r,0) = \hat{H}(x;r)$

In $\mathbb{R}_+ \times (0, T)$

$$\hat{U}_{tt} = rU_{tt} = r\left(U_{rr} + \frac{2}{r}U_r\right)$$

$$= rU_{rr} + 2U_r = (U + rU_r)_r$$

$$= \hat{U}_{rr}.$$

By the d'Alembert formula

$$\hat{U}(x; r, t) = \frac{1}{2}\left(\hat{G}(x; r+t) - \hat{G}(x; r-t)\right) + \frac{1}{2}\int_{-r+t}^{r+t} \hat{H}(x; y)\,dy$$

for $0 \leq r \leq t$.

Since $u(x, t) = \lim_{r \to 0^+} U(x; r, t)$.

$$u(x, t) = \lim_{r \to 0} \frac{1}{r}\hat{U}(x; r, t)$$

$$= \hat{G}'(x; t) + \hat{H}(x; t)$$

$$\therefore u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B_t(x)} g(\sigma)\, d\sigma \right) + t \fint_{\partial B_t(x)} h(\sigma)\, d\sigma .$$

$$= \fint_{\partial B_t(x)} t h(\sigma) + g(\sigma)\, d\sigma + t \frac{\partial}{\partial t} \fint_{\partial B_t(x)} g(\sigma)\, d\sigma)$$

$$\fint_{\partial B_t(x)} g = \fint_{\partial B_1(0)} g(x+t\sigma)\, d\sigma$$

$$\Rightarrow \frac{\partial}{\partial t} \fint_{\partial B_t(x)} g = \fint_{\partial B_1(0)} \nabla g(x+t\hat{\sigma}) \cdot \hat{\sigma}\, d\hat{\sigma}$$

$$= \fint_{\partial B_t(x)} \nabla g(\sigma) \frac{\sigma - x}{t}\, d\sigma$$

$$\therefore u(x,t)$$

$$= \fint_{\partial B_t(x)} t h(\sigma) + g(\sigma) + \nabla g(\sigma) \cdot (\sigma - x)\, d\sigma$$

Thm) Poisson's formula.

$(n=2)$. $g \in C^3(\mathbb{R}^2)$. $h \in C^2(\mathbb{R}^2)$. Then

$$u(x,t) = \frac{1}{2} \fint_{B_t(x)} \frac{t g(y) + t^2 h(y) + t \nabla g(y) \cdot (y-x)}{(t^2 - \|y-x\|^2)^{1/2}} \, dy$$

is the unique solution of class C^2 to the wave eq in $\mathbb{R}^2 \times [0,T)$

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pf) we define $\bar{u} : \mathbb{R}^3 \times [0,T) \to \mathbb{R}$ by

$$\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$$

Then. $\bar{u}_{tt} = \bar{u}_{xx}$ in $\mathbb{R}^3 \times [0,T)$

$\bar{u}(x,0) = \bar{g}(x)$. $\bar{u}_t(x,0) = \bar{h}(x)$ on $\mathbb{R}^3$.

where. $\bar{g}(x_1, x_2, x_3) = g(x_1, x_2)$

$\bar{h}(x_1, x_2, x_3) = h(x_1, x_2)$

Let $x=(x_1,x_2)\in\mathbb{R}^2$, $\bar{x}=(x_1,x_2,0)\in\mathbb{R}^3$

$B_t(x)\subseteq\mathbb{R}^2$. $\bar{B}_t(\bar{x})\subset\mathbb{R}^3$.

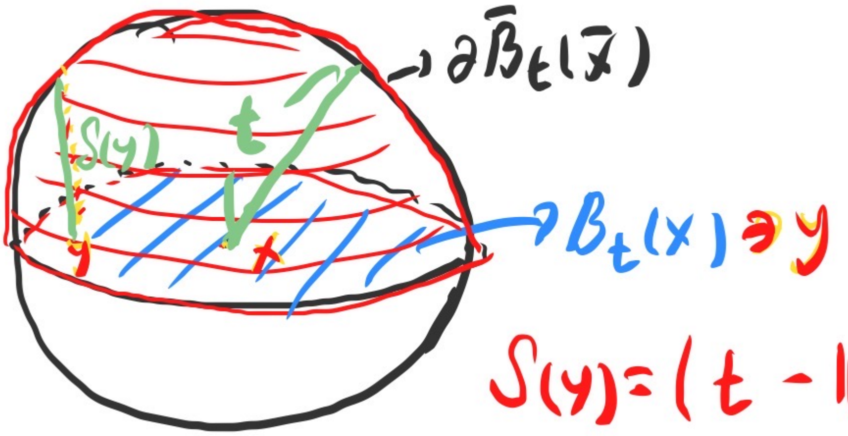
By Kirchhoff formula

$$u(x,t)=\bar{u}(\bar{x},t)$$
$$=\frac{\partial}{\partial t}\left(t\fint_{\partial\bar{B}_t(\bar{x})}\bar{g}\right)+t\fint_{\partial\bar{B}_t(\bar{x})}\bar{h}.$$

we have

$$\fint_{\partial\bar{B}_t(\bar{x})}\bar{g}=\frac{1}{4\pi t^2}\int_{\partial\bar{B}_t(\bar{x})}\bar{g}(\sigma)\,d\sigma$$
$$=\frac{2}{4\pi t^2}\int_{B_t(x)}g(y)\left(1+|\nabla S(y)|^2\right)^{1/2}dy.$$

where $S(y)=(t^2-\|y-x\|^2)^{1/2}$.



$S(y) = (t - \|y-x\|^2)^{1/2}$

$S : B_t(x) \to \mathbb{R}$.

the upper hemisphere of $\partial \bar{B}_t(\bar{x})$ is the graph of $S(y)$.

$\Rightarrow d\sigma = dS = (1 + \|\nabla S\|^2)^{1/2} dy$.

$$\nabla S = \frac{-(y-x)}{(t^2 - \|y-x\|^2)^{1/2}}$$

$$1 + \|\nabla S\|^2 = 1 + \frac{\|y-x\|^2}{t^2 - \|y-x\|^2} = \frac{t^2}{t^2 - \|y-x\|^2}$$

$\Rightarrow d\sigma = t\,(t - \|y-x\|^2)^{-1/2} dy$.

$$\Rightarrow \int_{\partial \bar{B}_t(\bar{x})} \bar{g}\, d\sigma = 2 \int_{B_t(x)} g(y)\, t\, (t - \|y-x\|^2)^{-1/2} dy.$$

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi} \int_{B_t(x)} g(y)\,(t^2 - \|y-x\|^2)^{-1/2}\, dy \right)$$

$$+ \frac{1}{2\pi} \int_{B_t(x)} h(y)\,(t^2 - \|y-x\|^2)^{-1/2}\, dy.$$

$$= \frac{1}{2} \frac{\partial}{\partial t}\left(t^2 \underbrace{\fint_{B_t(x)} g(y)\,(t^2 - \|y-x\|^2)^{-1/2}}_{I}\, dy \right)$$

$$+ \frac{1}{2} \fint_{B_t(x)} t^2 h(y)\,(t^2 - \|y-x\|^2)^{-1/2}\, dy.$$

$$I = t \fint_{B_1(0)} g(x+tz)\,(1 - \|z\|^2)^{-1/2}\, dz.$$

$$\frac{\partial}{\partial t} I = \fint_{B_1(0)} \frac{g(x+tz)}{(1-\|z\|^2)^{1/2}}\, dz$$

$$+ t \fint_{B_1(0)} \frac{z \cdot \nabla g(x+tz)}{(1-\|z\|^2)^{1/2}}\, dz.$$

$$= \fint_{B_t(x)} \frac{t\, g(y)}{(t^2 - \|y-x\|^2)^{1/2}}\, dy + t \fint_{B_t(x)} \frac{(y-x)\cdot \nabla g(y)}{(t^2 - \|y-x\|^2)}\, dy$$

$\blacksquare$

$n = 2k+1, \quad k \geq 1$

$$\hat{U} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} U(x;r,t)\right)$$

$$\hat{G} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} G(x;r)\right)$$

$$\hat{H} = \quad \text{''} \quad \left(\text{''} \quad H(x;r)\right)$$

$$\left[\text{ex. } \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 w = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{w_r}{r}\right) = \frac{1}{r}\left(\frac{w_{rr}}{r} - \frac{w_r}{r^2}\right) \right]$$

$$\Rightarrow \quad \hat{U}_{tt} = \hat{U}_{xx} \quad \text{in } \mathbb{R}_+ \times [0,T)$$

$$\hat{U} = \hat{G}, \quad \hat{U}_t = \hat{H} \quad \text{on } \mathbb{R}_+ \times \{0\}$$

$$\hat{U} = 0 \quad \text{on } r=0, \ t \in [0,T)$$

$$\Rightarrow u(x,t) = \gamma_k^{-1}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1} ⨍_{\partial B_t(x)} g\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1} ⨍_{\partial B_t(x)} h\right)\right]$$

where. $\gamma_k = 1\cdot 3\cdot 5 \cdots (2k-1)$.

$n = 2k$.

$$u(x,t) = \beta_k^{-1}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k}\int_{B_t(x)}\frac{g(y)}{(t^2-\|x-y\|^2)^{1/2}}\,dy\right)\right.$$
$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k}\int_{B_t(x)}\frac{h(y)}{(t^2-\|y-x\|^2)^{1/2}}\,dy\right)\right]$$

where $\beta_k = 2\cdot 4\cdot \cdots \cdot 2k = 2^k k!$.